

An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise

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Abstract

In this paper, we develop a new general approach to the existence and uniqueness theory of infinite dimensional stochastic equations of the form $dX + A(t)Xdt = XdW$ in $(0, T) \times H$, where $A(t)$ is a nonlinear monotone and demicontinuous operator from V to V' , coercive and with polynomial growth. Here, V is a reflexive Banach space continuously and densely embedded in a Hilbert space H of (generalized) functions on a domain $\mathcal{O} \subset \mathbb{R}^d$ and V' is the dual of V in the duality induced by H as pivot space. Furthermore, W is a Wiener process in H . The new approach is based on an operatorial reformulation of the stochastic equation which is quite robust under perturbation of $A(t)$. This leads to new existence and uniqueness results of a larger class of equations with linear multiplicative noise than the one treatable by the known approaches. In addition, we obtain regularity results for the solutions with respect to both the time and spatial variable which are sharper than the classical ones. New applications include stochastic partial differential equations, as e.g. stochastic transport equations.

Keywords: Maximal monotone operator, Stochastic integral, Operatorial equations, Schrödinger equation.

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1 Introduction

This work is concerned with a new general functional approach to the existence and uniqueness theory for nonlinear stochastic infinite dimensional equations with monotone and demicontinuous time dependent nonlinearities from a reflexive Banach space to its dual. Theorem 3.1 is the main general result obtained in this way, but the main point is the new method for its proof, namely rewriting the stochastic equation in operatorial form, which in turn is quite robust under perturbation, as formulated in Proposition 4.4. The latter then applies to a larger variety of examples, which do not exactly fit the general framework of Theorem 3.1, but can be treated by a direct approach on the basis of Proposition 4.4 (see, e.g., Section 6.3). In the literature on infinite dimensional stochastic differential equations, the type of equations in Theorem 3.1 was studied firstly by the classical Galerkin method, combined with monotonicity arguments by Pardoux [17] and developed later in a general setting by Krylov and Rozovski [13]. (A detailed presentation of these results is given in the monograph [18].) The approach we are proposing here is, principally, different and covers more general types of nonlinear stochastic PDEs, as long as the noise is linear multiplicative. Moreover, the results one obtains in this way are sharper, in regard to new regularity properties of the solutions. In a few words, it consists in representing, via a rescaling transformation, the stochastic initial value problem as a random operator equation of monotone type in a new convenient space of stochastic processes and in invoking the standard perturbation theory for nonlinear maximal monotone operators to get existence and uniqueness of solutions. In the special case, where the nonlinear operator is the subgradient of a convex function, the problem reduces to a convex optimization problem.

2 Preliminaries

Here we consider the stochastic differential equation

$$\begin{aligned} dX(t) + A(t)X(t)dt &= X(t)dW(t), \quad t \in (0, T), \\ X(0) &= x, \end{aligned} \tag{2.1}$$

in a real separable Hilbert space H , whose elements are functions or distributions on a bounded and open set $\mathcal{O} \subset \mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$. In particular, H can be any of the spaces $L^2(\mathcal{O})$, $H_0^1(\mathcal{O})$, $H^{-1}(\mathcal{O})$, $H^k(\mathcal{O})$,

$k = 1, \dots$, with the corresponding Hilbertian structure. Here $H_0^1(\mathcal{O})$, $H^k(\mathcal{O})$ are the standard L^2 -Sobolev spaces on \mathcal{O} , and W is a Wiener process of the form

$$W(t, \xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi) \beta_j(t), \quad \xi \in \mathcal{O}, \quad t \geq 0, \quad (2.2)$$

where $\{\beta_j\}_{j=1}^m$ is an independent system of real-valued Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$. Here, $e_j \in C^2(\overline{\mathcal{O}}) \cap H$ is an orthonormal basis in H , $\mu_j \in \mathbb{R}$, $j = 1, 2, \dots$.

The following hypotheses will be in effect throughout this work.

- (i) *There is a reflexive Banach space V with dual V' such that $V \subset H$, continuously and densely. Hence $V \subset H (\equiv H') \subset V'$ continuously and densely. (Note that this implies that also V is separable.) Moreover, V and V' are strictly convex (which can always be achieved by considering an appropriate equivalent norm on V by Asplund's Theorem, see [2, Theorem 1.2, p.2]).*
- (ii) *$A : [0, T] \times V \times \Omega \rightarrow V'$ is progressively measurable, i.e., for every $t \in [0, T]$, this operator restricted to $[0, t] \times V \times \Omega$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$ measurable.*
- (iii) *There is $\delta \geq 0$ such that, for each $t \in [0, T]$, $\omega \in \Omega$, the operator $u \rightarrow \delta u + A(t)u$ is monotone and demicontinuous (that is, strongly-weakly continuous) from V to V' . Moreover, there are $1 < p < \infty$, α_i and $\gamma_i \in \mathbb{R}$, $\alpha_1 > 0$, $i = 1, 2, 3$, such that, \mathbb{P} -a.s.,*

$$\langle A(t)u, u \rangle \geq \alpha_1 |u|_V^p + \alpha_2 |u|_H^2 + \alpha_3, \quad \forall u \in V, \quad t \in [0, T], \quad (2.3)$$

$$|A(t)u|_{V'} \leq \gamma_1 |u|_V^{p-1} + \gamma_2 + \gamma_3 |u|_H, \quad \forall u \in V, \quad t \in [0, T]. \quad (2.4)$$

- (iv) *$e^{\pm W} \in L^r((0, T) \times \Omega; V)$ for all $r \in [1, \infty)$ and $e^{\pm W(t)}$ is, for each t , a multiplier in V and a symmetric multiplier in H , such that there exists an (\mathcal{F}_t) -adapted, \mathbb{R}_+ -valued process $Z(t)$, $t \in [0, T]$, with $\sup_{t \in [0, T]} Z(t) < \infty$,*

\mathbb{P} -a.s., and such that \mathbb{P} -a.s.

$$\begin{aligned} |e^{\pm W(t)} y|_V &\leq Z(t) |y|_V, \quad \forall t \in [0, T], \quad \forall y \in V, \\ |e^{\pm W(t)} y|_H &\leq Z(t) |y|_H, \quad \forall t \in [0, T], \quad \forall y \in H. \end{aligned} \quad (2.5)$$

Furthermore, we assume that, \mathbb{P} -a.s.,

$$\begin{aligned} \langle e^{\pm W(t)}x, y \rangle &= \langle x, e^{\pm W(t)}y \rangle, \quad \forall x, y \in H, \quad t \in [0, T], \\ t \longmapsto e^{\pm W(t)} &\in H \text{ is continuous.} \end{aligned} \quad (2.6)$$

We also note that, by Fernique's theorem,

$$\exp \left(\sup_{0 \leq t \leq T} |W(t)|_\infty \right) \in L^q(\Omega), \quad \forall q \in (0, \infty). \quad (2.7)$$

which will be used to estimate $\mathbb{E}|e^{W(t)}|_V^q$ and $\mathbb{E}|e^{W(t)}|_H^q$ to verify (iv) in many situations, where V is a subspace of L^q , $1 < q < \infty$, or a Sobolev space on $\mathcal{O} \subset \mathbb{R}^d$. Above and below, $|\cdot|_V$ and $|\cdot|_{V'}$ denote the norms of V and V' , respectively, and $\langle \cdot, \cdot \rangle$ the duality pairing between V and V' ; on $H \times H$, $\langle \cdot, \cdot \rangle$ is just the scalar product of H . The norm of H is denoted by $|\cdot|_H$, and $\mathcal{B}(H), \mathcal{B}(V)$ etc. are used to denote the class of Borelian sets on the corresponding spaces.

As regards the basis $\{e_j\}_{j=1}^\infty$ in (2.2), we assume that there exist $\tilde{\gamma}_j \in [1, \infty)$ such that

$$\begin{aligned} |ye_j|_H &\leq \tilde{\gamma}_j |e_j|_\infty |y|_H, \quad \forall y \in H, \quad j = 1, 2, \dots, \\ \nu &:= \sum_{j=1}^\infty \mu_j^2 \tilde{\gamma}_j^2 |e_j|_\infty^2 < \infty, \end{aligned} \quad (2.8)$$

and, for

$$\mu := \frac{1}{2} \sum_{j=1}^\infty \mu_j^2 e_j^2, \quad (2.9)$$

we assume that μ is a multiplier in V and a symmetric multiplier in H . All these assumptions on W , as well as Hypothesis (iv), typically hold in applications, as we will see in the examples in Section 6.

For $y \in H$, we define the operator

$$\sigma(y)h = \sum_{j=1}^\infty \mu_j y \langle h, e_j \rangle e_j, \quad h \in H.$$

Then, by (2.8), we see that σ is linear continuous form H to the space $L_2(H)$ of all Hilbert-Schmidt operators on H and $X dW = \sigma(X) d\widetilde{W}$ in the notation

of e.g. [12], [18] with \widetilde{W} being the cylindrical Wiener process on H , informally written as $\widetilde{W}(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j$.

As usual, we set $p' := \frac{p}{p-1}$, $1 < p < \infty$.

Definition 2.1 By a *solution* to (2.1) for $x \in H$, we mean an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $X : [0, T] \rightarrow H$ with continuous sample paths, which satisfies

$$X \in L^\infty(0, T; L^2(\Omega; H)) \quad (2.10)$$

$$X(t) + \int_0^t A(s)X(s)ds = x + \int_0^t X(s)dW(s), \quad t \in [0, T], \quad (2.11)$$

$$AX \in L^{p'}((0, T) \times \Omega; V'), \quad X \in L^p((0, T) \times \Omega; V). \quad (2.12)$$

The stochastic integral arising in (2.11) is considered in Itô's sense.

Under Hypotheses (i)–(iii), equation (2.1) was studied in [17] and [13]. The present approach is a general unifying one and leads to new regularity results for (2.1) and, in particular, implies that the solution X to (2.1) is absolutely continuous in t up to multiplication with $e^{-W(t)}$. It should be said, however, that it applies only to stochastic differential equations which admit a formulation in the variational setting $V \subset H \subset V'$ and with coercive nonlinearities. In a few words, the method is the following. By the rescaling transformation

$$X(t) = e^{W(t)}y(t), \quad t \geq 0, \quad (2.13)$$

one formally reduces equation (2.1) to the random differential equation

$$\begin{aligned} \frac{dy}{dt}(t) + e^{-W(t)}A(t)(e^{W(t)}y(t)) + \mu y(t) &= 0, \quad \text{a.e. } t \in (0, T), \\ y(0) &= x. \end{aligned} \quad (2.14)$$

The random Cauchy problem (2.14) will be treated as an operatorial equation in a convenient Hilbert space of stochastic processes to be described later on. A nice feature of the approach to be developed below is that, though (2.14) is not an equation of monotone type, it can be rewritten as an operator equation of monotone type in an appropriate space of infinite dimensional stochastic processes on $[0, T]$. It should be emphasized that the rescaling approach to the treatment of stochastic PDE with linearly multiplicative noise was previously applied in [6], [7] and one of the main advantages of this approach is that it leads to sharp pointwise estimates and new pathwise

regularity for solutions to (2.1). It should be mentioned, however, that such a result cannot be proved for equations with more general Gaussian processes $\sigma(X)W$, the linearity of σ being essential for this approach.

Notations. $H^k(\mathcal{O})$, $k = 1, 2$, and $H_0^1(\mathcal{O})$, $W_0^{1,p}(\mathcal{O})$, $W^{-1,p'}(\mathcal{O})$, $1 \leq p \leq \infty$, are Sobolev spaces on \mathcal{O} (see, e.g., [1], [10]). If U is a Banach space we denote by $L^p(0, T; U)$, $1 \leq p \leq \infty$, the space of all L^p -integrable U -valued functions on $(0, T)$. Similarly, the space $L^p((0, T) \times \Omega; U)$ is defined. By $W^{1,p}([0, T]; U)$ we denote the space of absolutely continuous functions $y : [0, T] \rightarrow U$ such that $\frac{dy}{dt} \in L^p(0, T; V)$. In the following we refer to [2] for notations and standard results on the theory of maximal monotone operators in Banach spaces.

3 The main result

Theorem 3.1 *Under Hypotheses (i)–(iv), for each $x \in H$, equation (2.1) has a unique solution X (in the sense of Definition 2.1). Moreover, the function $t \rightarrow e^{-W(t)}X(t)$ is V' -absolutely continuous on $[0, T]$ and*

$$\mathbb{E} \int_0^T \left| e^{W(t)} \frac{d}{dt}(e^{-W(t)}X(t)) \right|_{V'}^{p'} dt < \infty. \quad (3.1)$$

As mentioned earlier, the proof strategy is to reduce (2.1) via the transformation (2.13) to the random differential equation (2.14), which will be treated afterwards as a deterministic evolution equation.

A heuristic application of the Itô formula in (2.1) leads to (see Lemma 8.1 in the Appendix for its rigorous justification)

$$dX = e^W dy + e^W y dW + \mu e^W y dt, \quad (3.2)$$

where y is given by (2.13). Substituting into (2.1) yields (2.14), that is,

$$\begin{aligned} \frac{dy}{dt} + e^{-W} A(t)(e^W y) + \mu y &= 0, \quad t \in (0, T), \\ y(0) &= x. \end{aligned} \quad (3.3)$$

Definition 3.2 A solution to (3.3) is an H -valued $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $y = y(t)$, $t \in [0, T]$, with continuous sample paths satisfying the following conditions

$$\sup_{t \in [0, T]} \mathbb{E} |e^{W(t)} y(t)|_H^2 < \infty, \quad (3.4)$$

$$\mathbb{E} \int_0^T \left| e^{W(t)} \frac{dy}{dt}(t) \right|_{V'}^{p'} dt < \infty, \quad (3.5)$$

$$\begin{aligned} \frac{dy}{dt}(t) + e^{-W(t)} A(t)(e^{W(t)} y(t)) + \mu y(t) &= 0, \text{ a.e. } t \in (0, T), \\ y(0) &= x, \end{aligned} \quad (3.6)$$

$$\mathbb{E} \int_0^T |e^{W(t)} y(t)|_{V'}^p dt < \infty, \quad (3.7)$$

(hence $A(t)(e^W y) \in L^{p'}(\Omega \times (0, T); V')$, by (2.4)).

In particular, it follows by (2.5) and (3.5), (3.7) that $y : [0, T] \rightarrow V'$ is absolutely continuous \mathbb{P} -a.s. and

$$y \in L^p(0, T; V), \quad \frac{dy}{dt} \in L^{p'}(0, T; V'), \quad \mathbb{P}\text{-a.s.} \quad (3.8)$$

We have

Proposition 3.3 *Under Hypotheses (i)–(iv), for each $x \in H$, equation (3.3) has a unique solution y .*

We shall prove Proposition 3.3 in Section 4 via an operatorial approach to be described later on in Section 4. Now, we reduce the proof of Theorem 3.1 to Proposition 3.3.

Proof of Theorem 3.1. By Lemma 8.1 in the Appendix, we know that equations (2.1) and (3.3) are equivalent via the rescaling transformation (2.13) and so existence and uniqueness of a solution X to (2.1) in the sense of Definition 2.1 follows by Proposition 3.3. As regards (3.1), this is a direct consequence of (3.5), (3.8). ■

Remark 3.4 We must emphasize that (3.1) is a new and somewhat surprising regularity result for the solution X to the stochastic equation (2.1). It amounts to saying that up to multiplication with e^{-W} , the process X is \mathbb{P} -a.s. absolutely continuous V' -valued on $[0, T]$.

4 An operatorial approach to equation (3.3)

Without loss of generality, we may assume that $A(t)$ satisfies the strong monotonicity condition

$$\langle A(t)u - A(t)v, u - v \rangle \geq \nu |u - v|_H^2, \quad \forall u, v \in V, \quad (4.1)$$

where ν is defined by (2.8). Indeed, it is easily seen that, by the substitution $y \rightarrow e^{-(\nu+\delta)t}y$, equation (3.3) can be equivalently written as

$$\begin{aligned} \frac{dy}{dt}(t) + e^{-W(t)}\tilde{A}(t)(e^{W(t)}y(t)) + \mu y(t) &= 0, \quad t \in (0, T), \\ y(0) &= x, \end{aligned}$$

where

$$\tilde{A}(t)y = e^{-(\delta+\nu)t}A(t)(e^{(\delta+\nu)t}y) + (\delta + \nu)y.$$

Then $\tilde{A} : V \rightarrow V'$ and \tilde{A} satisfies (ii), (iii) and (4.1).

We note that the operator $y \rightarrow e^{-W(t)}A(t)(e^{W(t)}y)$ is not monotone in $V \times V'$ and so the standard existence theory (see, e.g., [2], [15]) is not applicable in this case. Therefore, we define new spaces \mathcal{H} , \mathcal{V} and \mathcal{V}' , as follows. \mathcal{H} is the Hilbert space of all H -valued $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes $y : [0, T] \rightarrow H$ such that

$$|y|_{\mathcal{H}} = \left(\mathbb{E} \int_0^T |e^{W(t)}y(t)|_H^2 dt \right)^{\frac{1}{2}} < \infty, \quad (4.2)$$

where \mathbb{E} denotes the expectation in the above probability space. The space \mathcal{H} is endowed with the norm $|\cdot|_{\mathcal{H}}$ coming from the scalar product

$$\langle y, z \rangle_{\mathcal{H}} = \mathbb{E} \int_0^T \langle e^{W(t)}y(t), e^{W(t)}z(t) \rangle dt. \quad (4.3)$$

\mathcal{V} is the space of the V -valued $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $y : [0, T] \rightarrow V$ such that

$$|y|_{\mathcal{V}} = \left(\mathbb{E} \int_0^T |e^{W(t)}y(t)|_V^p dt \right)^{\frac{1}{p}} < \infty. \quad (4.4)$$

Clearly, \mathcal{V} is reflexive. \mathcal{V}' (the dual of \mathcal{V}) is the space of all V' -valued $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes $y : [0, T] \rightarrow V'$ such that

$$|y|_{\mathcal{V}'} = \left(\mathbb{E} \int_0^T |e^{W(t)}y(t)|_{V'}^{p'} dt \right)^{\frac{1}{p'}} < \infty, \quad (4.5)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. If $2 \leq p < \infty$, we have

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}' \quad (4.6)$$

continuously and densely and

$${}_{\mathcal{V}'} \langle u, v \rangle_{\mathcal{V}} = \mathbb{E} \int_0^T \langle e^{W(t)} u(t), e^{W(t)} v(t) \rangle dt, \quad v \in \mathcal{V}, \quad u \in \mathcal{V}', \quad (4.7)$$

is just the duality pairing between \mathcal{V} and \mathcal{V}' . We have also

$${}_{\mathcal{V}'} \langle u, v \rangle_{\mathcal{V}} = \langle u, v \rangle_{\mathcal{H}}, \quad \forall u \in \mathcal{H}, \quad v \in V. \quad (4.8)$$

In the case where $1 < p < 2$ we replace \mathcal{V} by $\mathcal{V} \cap \mathcal{H}$ and still have (4.6).

We also note that, by Hypothesis (iv), we have the continuous embeddings

$$L^{p_2}((0, T) \times \Omega; V) \subset \mathcal{V} \subset L^{p_1}((0, T) \times \Omega; V), \quad \forall 1 \leq p_1 < p, \quad \max(p, 2) < p_2. \quad (4.9)$$

Now, we fix $x \in H$ and define on \mathcal{V} the operators $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{V} \rightarrow \mathcal{V}'$ as follows:

$$(\mathcal{A}y)(t) = e^{-W(t)} A(t)(e^{W(t)} y(t)) - \nu y(t), \quad \text{a.e. } t \in (0, T), \quad y \in \mathcal{V}, \quad (4.10)$$

$$(\mathcal{B}y)(t) = \frac{dy}{dt}(t) + (\mu + \nu)y(t), \quad \text{a.e. } t \in (0, T), \quad y \in D(\mathcal{B}),$$

$$D(\mathcal{B}) = \left\{ y \in \mathcal{V} : y \in AC([0, T]; V') \cap C([0, T]; H), \quad \mathbb{P}\text{-a.s.}, \quad \frac{dy}{dt} \in \mathcal{V}', \quad y(0) = x \right\}. \quad (4.11)$$

Here, $AC([0, T]; V')$ is the space of all absolutely continuous V' -valued functions on $[0, T]$. The fact that indeed $\mathcal{A}(\mathcal{V}) \subset \mathcal{V}'$ follows by (2.4) since $p \geq p'$ if $p \geq 2$, and \mathcal{V} is replaced by $\mathcal{V} \cap \mathcal{H}$ for $1 < p < 2$.

We also note that, if $p \geq 2$ and $y \in L^p(0, T; V)$ and $\frac{dy}{dt} \in L^{p'}(0, T; V')$, that is, if $y \in W^{1,p'}([0, T]; V') \cap L^p(0, T; V)$, then $y \in C([0, T]; H)$ and $\frac{dy}{dt}$ is just the derivative of y in the sense of V' -valued distributions on $(0, T)$ (see, e.g., [2, Corollary 2.1, p. 33]) and so the condition $y \in C([0, T]; H)$ in the definition of $D(\mathcal{B})$ is redundant. That the same is true for $1 < p < 2$, follows by Lemma 8.2 in the Appendix.

We note also that, by virtue of (4.9), we have

$$\frac{dy}{dt} \in L^{p_1}((0, T) \times \Omega; V'), \quad \forall y \in D(\mathcal{B}),$$

for any $1 \leq p_1 < p'$ and so $y : [0, T] \rightarrow L^{p_1}(\Omega; V')$ is absolutely continuous, \mathbb{P} -a.s.

Then, the Cauchy problem (3.3) can be written as the operatorial equation

$$\mathcal{B}y + \mathcal{A}y = 0. \quad (4.12)$$

We have

Lemma 4.1 *The operators \mathcal{A} and \mathcal{B} are maximal monotone from \mathcal{V} to \mathcal{V}' .*

Proof. $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ is maximal monotone by virtue of the classical Minty-Browder result (see, e.g., [2], p. 43) because, as easily seen, by virtue of (ii)–(iv) and (4.1) it is monotone, that is,

$${}_{\mathcal{V}'} \langle \mathcal{A}y - \mathcal{A}z, y - z \rangle_{\mathcal{V}} \geq 0, \quad \forall y, z \in \mathcal{V},$$

and demicontinuous on \mathcal{V} (that is strongly-weakly continuous).

Indeed, if $y_n \rightarrow y$ in \mathcal{V} , then Hypothesis (iii) implies that (along a subsequence) $y_n \rightarrow y$ $dt \otimes \mathbb{P}$ -a.e. Hence $dt \otimes \mathbb{P}$ -a.e.

$$\begin{aligned} (\mathcal{A}y_n)(t) &= e^{-W(t)}A(t)(e^{W(t)}y_n(t)) - \nu y_n(t) \\ &\longrightarrow e^{-W(t)}A(t)(e^{W(t)}y(t)) - \nu y(t) \quad \text{weakly in } V' \end{aligned} \quad (4.13)$$

(because $e^{W(t)}$ is a multiplier in V). We also have by (2.4)

$$|\mathcal{A}y_n|_{\mathcal{V}'} \leq \gamma_1 |y_n|_{\mathcal{V}}^{p-1} + \gamma_2 + \gamma_3 |y_n|_{\mathcal{H}}, \quad \forall n \in \mathbb{N},$$

and so, by the Banach-Alaoglu theorem, we have that (along a subsequence)

$$\mathcal{A}y_n \rightarrow \eta \quad \text{weakly in } \mathcal{V}'. \quad (4.14)$$

On the other hand, by (4.13) we have that, for a countable dense subset of $\psi \in V$,

$$\begin{aligned} {}_{\mathcal{V}'} \langle \mathcal{A}y_n(t), \psi \rangle_V &\rightarrow {}_{\mathcal{V}'} \langle e^{-W(t)}A(t)(e^{W(t)}y(t)) - \nu y(t), \psi \rangle_V, \\ &dt \otimes \mathbb{P} - \text{a.e. on } (0, T) \times \Omega. \end{aligned} \quad (4.15)$$

By (4.14) and (4.15), we see that

$$\eta = e^{-W(t)}A(t)(e^{W(t)}y(t)) - \nu y(t) = (\mathcal{A}y)(t), \quad dt \otimes \mathbb{P} - \text{a.e. in } (0, T) \times \Omega,$$

as claimed.

We prove, now, that \mathcal{B} is maximal monotone. To prove the monotonicity, let $y_1, y_2 \in D(\mathcal{B})$ and set $y := y_1 - y_2$. Then

$$\begin{aligned} \nu' \langle \mathcal{B}(y), y \rangle_{\mathcal{V}} &= \mathbb{E} \int_0^T \left\langle e^{W(t)} \frac{d}{dt} y(t), e^{W(t)}(y(t)) \right\rangle dt \\ &\quad + \mathbb{E} \int_0^T \langle (\mu + \nu) y e^W, y e^W \rangle dt. \end{aligned} \quad (4.16)$$

By Lemma 8.1(jjj) in the Appendix, we have, for $y \in D(\mathcal{B})$,

$$d(ye^W) = e^W dy + e^W y dW + \mu e^W y dt, \quad (4.17)$$

and, applying Itô's formula to $|ye^W|_H^2$ (which is justified because, e.g., if $p \geq 2$, Theorem 4.2.5 in [18] and, if $p \in \{1, 2\}$, Remark 4.2.8 (iii) in [16], respectively, can be applied due to Lemma 8.1 (jjj) below), we obtain

$$\begin{aligned} \frac{1}{2} d|e^W y|_H^2 &= \left\langle e^W \frac{dy}{dt}, e^W y \right\rangle dt + \langle e^W y, e^W y dW \rangle \\ &\quad + \langle \mu e^W y, e^W y \rangle dt + \frac{1}{2} \sum_{j=1}^{\infty} |e^W y e_j|_H^2 \mu_j^2 dt. \end{aligned} \quad (4.18)$$

Then, by (2.8), we obtain that

$$\begin{aligned} &\mathbb{E} \int_0^T \left\langle e^W \frac{dy}{dt}, e^W y \right\rangle dt \\ &\geq -\mathbb{E} \int_0^T \langle (\mu + \nu) y e^W, y e^W \rangle dt + \frac{1}{2} \mathbb{E} |e^{W(T)} y(T)|_H^2 - \frac{1}{2} |y(0)|_H^2. \end{aligned} \quad (4.19)$$

Substituting into (4.16) yields (because $y(0) = 0$)

$$\nu' \langle \mathcal{B}y, y \rangle_{\mathcal{V}} \geq \frac{1}{2} \mathbb{E} |e^{W(T)} y(T)|_H^2. \quad (4.20)$$

Hence, \mathcal{B} is monotone. ■

As regards the maximality of \mathcal{B} in $\mathcal{V} \times \mathcal{V}'$, we have

Lemma 4.2 *Let $\mathcal{B} : \mathcal{V} \rightarrow \mathcal{V}'$ be the operator defined by (4.11). Then, \mathcal{B} is maximal monotone in $\mathcal{V} \times \mathcal{V}'$.*

Proof. We denote by $J : V \rightarrow V'$ the duality mapping of V . We recall that $|J(y)|_{V'} = |y|_V$, ${}_{V'}\langle J(y), y \rangle_V = |y|_V^2$. Moreover, since, by assumption, V' is strictly convex, then J is single valued, monotone and demicontinuous (see [2], p. 12). We define the mapping $F : \mathcal{V} \rightarrow \mathcal{V}'$ as

$$(Fy)(t) = e^{-W(t)} J(e^{W(t)} y(t)) |e^{W(t)} y(t)|_V^{p-2}, \quad \forall t \in (0, T), \quad y \in \mathcal{V}. \quad (4.21)$$

Clearly, F is monotone and demicontinuous from \mathcal{V} to \mathcal{V}' . Furthermore, $y \rightarrow F(y)|y|_{\mathcal{V}}^{2-p}$ is just the duality mapping of \mathcal{V} , if $p \geq 2$. If $p \in (1, 2)$, we replace F by $F + I$ and proceed analogously as below with $\nu + 1$ replacing ν .

It suffices to show that, for each $f \in \mathcal{V}'$, the equation $\mathcal{B}y + F(y) = f$, where F is given by (4.21), has a solution y (see [2, Theorem 2.3, p. 35]).

Equivalently, $y \in D(\mathcal{B})$, and

$$\begin{aligned} \frac{dy}{dt} + e^{-W} J(e^W y) |e^W y|_V^{p-2} + (\mu + \nu)y &= f, \quad t \in (0, T), \\ y(0) &= x. \end{aligned} \quad (4.22)$$

We set $G(z) = J(z)|z|_V^{p-2}$, $\forall z \in V$ and note that G is monotone, demicontinuous and coercive from V to V' . We denote by G_H the restriction of G to H , that is, G_H is the operator with graph $\{(u, Gu) : u \in V\} \cap V \times H$. The operator G_H is maximal monotone in $H \times H$. Let G_λ denote its Yosida approximation, that is,

$$G_\lambda(z) = G(I + \lambda G_H)^{-1}(z) = \frac{1}{\lambda} (z - (I + \lambda G_H)^{-1}(z)), \quad \forall z \in H. \quad (4.23)$$

We consider now the approximating equation of (4.22)

$$\begin{aligned} \frac{dy_\lambda}{dt} + e^{-W} G_\lambda(e^W y_\lambda) + (\mu + \nu)y_\lambda &= f, \quad t \in [0, T], \\ y_\lambda(0) &= x. \end{aligned} \quad (4.24)$$

We assume first that $f \in \mathcal{H}$ and prove later on that the existence extends to all of \mathcal{V}' . Since G_λ is Lipschitz on H , it follows that (4.23) has a unique solution $y_\lambda \in C([0, T]; H)$, \mathbb{P} -a.s. Moreover, $t \rightarrow y_\lambda(t)$ is $(\mathcal{F}_t)_{t \geq 0}$ -adapted and by (4.18)

$$\begin{aligned}
& \frac{1}{2} \mathbb{E} |e^{W(t)} y_\lambda(t)|_H^2 + \mathbb{E} \int_0^t \langle G_\lambda(e^W y_\lambda), e^W y_\lambda \rangle ds + \nu \mathbb{E} \int_0^t |e^W y_\lambda|_H^2 ds \\
&= \frac{1}{2} \mathbb{E} \int_0^t \sum_{j=1}^\infty |e^W y_\lambda e_j|_H^2 \mu_j^2 ds + \mathbb{E} \int_0^t \langle f e^W, e^W y_\lambda \rangle ds + \frac{1}{2} |x|_H^2, \\
& \qquad \qquad \qquad \forall t \in [0, T].
\end{aligned} \tag{4.25}$$

Taking into account that

$$\begin{aligned}
\langle G_\lambda(z), z \rangle &\geq (G((I + \lambda G_H)^{-1}(z)), (I + \lambda G_H)^{-1}(z)) \\
&= |(I + \lambda G_H)^{-1}(z)|_{V'}^p, \quad \forall z \in H,
\end{aligned} \tag{4.26}$$

we get, via Gronwall's lemma, for all $t \in [0, T]$,

$$\mathbb{E} |e^{W(t)} y_\lambda(t)|_H^2 + \mathbb{E} \int_0^t |(I + \lambda G_H)^{-1}(e^W y_\lambda(s))|_{V'}^p ds \leq C(|f|_H^2 + |x|_H^2). \tag{4.27}$$

Hence, along a subsequence $\{\lambda\} \rightarrow 0$, we have

$$\begin{aligned}
e^W y_\lambda &\longrightarrow e^W y \quad \text{weak* in } L^\infty(0, T; L^2(\Omega; H)), \\
(I + \lambda G_H)^{-1}(e^W y_\lambda) &\longrightarrow z \quad \text{weakly in } L^p((0, T) \times \Omega; V), \\
G_\lambda(e^W y_\lambda) &\longrightarrow \eta \quad \text{weakly in } L^{p'}((0, T) \times \Omega; V'),
\end{aligned} \tag{4.28}$$

where the latter follows, since for $z \in H$, $\lambda > 0$,

$$|G_\lambda(z)|_{V'}^{p'} = |(I + \lambda G_H)^{-1}(z)|_{V'}^p. \tag{4.29}$$

(4.23) and (4.29) imply that

$$|e^W y_\lambda - (I + \lambda G_H)^{-1}(e^W y_\lambda)|_{V'}^{p'} \leq C_1 \lambda^{p'} |(I + \lambda G_H)^{-1}(e^W y_\lambda)|_{V'}^p.$$

Therefore, $z = e^W y$. Moreover, letting $\lambda \rightarrow 0$ in (4.24), we get \mathbb{P} -a.s.

$$\begin{aligned}
\frac{dy}{dt} + e^{-W} \eta + (\mu + \nu) y &= f \quad \text{a.e. } t \in (0, T), \\
y(0) &= x,
\end{aligned} \tag{4.30}$$

and, clearly, $y \in D(\mathcal{B})$. It remains to show that $\eta = G(e^W y)$. To this end, by the inequality in (4.26) and Lemma 8.3 below (see, also, [2, Lemma 2.3]), it suffices to show that

$$\limsup_{\lambda \rightarrow 0} \mathbb{E} \int_0^T \int_0^t \langle G_\lambda(e^W y_\lambda), e^W y_\lambda \rangle ds dt \leq \mathbb{E} \int_0^T \int_0^t \langle \eta, e^W y \rangle ds dt. \tag{4.31}$$

To this end, we note that, by (4.25) and (4.28), (4.29), we have

$$\begin{aligned}
& \limsup_{\lambda \rightarrow 0} \mathbb{E} \int_0^T \int_0^t \langle G_\lambda(e^W y_\lambda), e^W y_\lambda \rangle ds dt \\
& \leq -\nu \mathbb{E} \int_0^T \int_0^t |e^W y|_H^2 ds dt - \frac{1}{2} \mathbb{E} \int_0^t |e^{W(t)} y(t)|_H^2 dt + \frac{T}{2} |x|_H^2 \\
& + \mathbb{E} \int_0^T \int_0^t \langle f e^W, e^W y \rangle ds dt + \frac{1}{2} \mathbb{E} \sum_{j=1}^{\infty} \int_0^T \int_0^t |e^W y e_j|_H^2 \mu_j^2 ds dt,
\end{aligned}$$

because both $z \rightarrow \mathbb{E} \int_0^t |z|_H^2 dt$ and the function

$$z \rightarrow \mathbb{E} \int_0^T \int_0^t \left(\nu |z|_H^2 - \frac{1}{2} \sum_{j=1}^{\infty} |z e_j|_H^2 \mu_j^2 \right) ds dt$$

are continuous on $L^2((0, T) \times \Omega; H)$ and convex, hence weakly lower semi-continuous in $L^2((0, T) \times \Omega; H)$. On the other hand, (4.30), Lemma 8.1 (jjj) and (4.18) yield

$$\begin{aligned}
& \frac{1}{2} \int_0^T \mathbb{E} |e^W y(t)|_H^2 dt + \mathbb{E} \int_0^T \int_0^t \langle \eta, e^W y \rangle ds dt \\
& = \frac{T}{2} |x|^2 - \nu \mathbb{E} \int_0^T \int_0^t |e^W y|_H^2 ds dt \\
& + \frac{1}{2} \mathbb{E} \sum_{j=1}^{\infty} \int_0^T \int_0^t |e^W y e_j|_H^2 \mu_j^2 ds dt + \mathbb{E} \int_0^T \int_0^t \langle f e^W, e^W y \rangle ds dt,
\end{aligned} \tag{4.32}$$

and so, (4.31) follows.

Now, let $f \in \mathcal{V}'$ and choose $\{f_n\} \subset \mathcal{H}$ such that $f_n \rightarrow f$ strongly in \mathcal{V}' . If we denote by y_n the corresponding solution of (4.22), we obtain as in the previous case (see (4.25))

$$\begin{aligned}
& \frac{1}{2} \mathbb{E} |e^{W(t)} y_n(t)|_H^2 + \mathbb{E} \int_0^t \langle G(e^{W(s)} y_n(s)), e^{W(s)} y_n(s) \rangle ds + \nu \mathbb{E} \int_0^t |e^{W(s)} y_n(s)|_H^2 ds \\
& = \frac{1}{2} \mathbb{E} \int_0^t \sum_{j=1}^{\infty} |e^{W(s)} y_n(s) e_j|_H^2 \mu_j^2 ds + \mathbb{E} \int_0^t \langle f_n(s) e^{W(s)}, y_n(s) e^{W(s)} \rangle ds + \frac{1}{2} |x|_H^2,
\end{aligned}$$

and this yields as above

$$\mathbb{E}|e^{W(t)}y_n(t)|_H^2 + \mathbb{E} \int_0^t |e^{W(s)}y_n(s)|_V^p ds \leq C \left(|x|_H^2 + \int_0^t |f_n(s)e^{W(s)}|_{V'}^{p'} ds \right)$$

where C is independent of n . Hence $\{y_n\}_n$ is bounded in $\mathcal{V} \cap L^\infty((0, T) \times \Omega; H)$ and so, along a subsequence, we have

$$\begin{aligned} e^W y_n &\longrightarrow e^W y \quad \text{weak}^* \text{ in } L^\infty(0, T; L^2(\Omega; H)) \text{ and} \\ &\quad \text{weakly in } L^p((0, T) \times \Omega; V) \\ G(e^W y_n) &\longrightarrow \eta \quad \text{weakly in } L^{p'}((0, T) \otimes \Omega; V'), \end{aligned}$$

where y satisfies (4.30). Arguing as in the proof of (4.31), we see that

$$\limsup_{n \rightarrow \infty} \mathbb{E} \int_0^T \int_0^t \langle G(e^W y_n), e^W y_n \rangle ds dt \leq \mathbb{E} \int_0^T \int_0^t \langle \eta, e^W y \rangle ds dt,$$

and, by Lemma 8.3, this implies that $\eta = G(e^W y)$, and so y is a solution to (4.22), as claimed. This completes the proof. ■

Proof of Proposition 3.3. Since \mathcal{A}, \mathcal{B} are maximal monotone in $\mathcal{V} \times \mathcal{V}'$ and $D(\mathcal{A}) = \mathcal{V}$, we infer (see, e.g., [2], p. 43) that $\mathcal{A} + \mathcal{B}$ is maximal monotone in $\mathcal{V} \times \mathcal{V}'$. Hence, the equation

$$\lambda F(y_\lambda) + \mathcal{B}y_\lambda + \mathcal{A}y_\lambda = 0 \tag{4.33}$$

has, for each $\lambda > 0$, a unique solution $y_\lambda \in D(\mathcal{B})$. (See [2], p. 35.)

We may rewrite (4.33) as

$$\begin{aligned} \lambda J(e^{W(t)}y_\lambda(t))|e^{W(t)}y_\lambda(t)|_V^{p-2} + e^{W(t)} \frac{dy_\lambda(t)}{dt} + A(t)(e^{W(t)}y_\lambda(t)) \\ + \mu e^{W(t)}y_\lambda(t) = 0, \quad t \in (0, T), \quad \omega \in \Omega, \\ y_\lambda(0) = x. \end{aligned}$$

If we apply $\langle e^{W(t)}y_\lambda(t), \cdot \rangle$ to the latter and integrate over $(0, T) \times \Omega$, we get by (2.3) and (4.19) that

$$\begin{aligned} (\alpha_1 + \lambda) \mathbb{E} \int_0^t |e^{W(s)}y_\lambda(s)|_V^p ds + \frac{1}{2} \mathbb{E}|e^{W(t)}y_\lambda(t)|_H^2 \\ \leq \frac{1}{2} |x|_H^2 + (|\alpha_2| + \nu) \int_0^t \mathbb{E}|e^{W(s)}y_\lambda(s)|_H^2 ds, \quad \forall t \in [0, T]. \end{aligned}$$

By Gronwall's lemma, we see that

$$|y_\lambda|_{\mathcal{V}} \leq C, \quad \forall \lambda > 0,$$

where C is independent of λ and so, along a subsequence again denoted by λ , we have

$$\lambda F(y_\lambda) \rightarrow 0 \text{ strongly in } \mathcal{V}', \quad y_\lambda \rightarrow y^* \text{ weakly in } \mathcal{V} \text{ as } \lambda \rightarrow 0.$$

Hence, $\mathcal{A}y_\lambda + \mathcal{B}y_\lambda \rightarrow 0$ in \mathcal{V}' and, since $\mathcal{A} + \mathcal{B}$ is weakly-strongly closed (as a consequence of the maximal monotonicity), we have that $y^* \in D(\mathcal{B})$ and

$$\mathcal{B}y^* + \mathcal{A}y^* = 0.$$

So, y^* is a solution to (3.3) in sense of Definition 3.2. If y and z are two solutions, we have by the monotonicity of \mathcal{A} and (4.20) that

$$y(T) = z(T), \quad \mathbb{P}\text{-a.s.}$$

On the other hand, y, z are solutions to (3.3) on each interval $(0, t)$ and so we conclude, by the H -continuity of y and z , that \mathbb{P} -a.s.,

$$y(t) = z(t), \quad \forall t \in [0, T].$$

This completes the proof.

Remark 4.3 The assumption:

$A(t)$ single-valued and demicontinuous from V to V'

can be relaxed to:

For each $t \in (0, T)$, $A(t)$ is a maximal monotone (multivalued) operator from V to V' such that $D(A(t))=V$, $\forall t \in [0, T]$ and $A(t)(e^W y) \cap \mathcal{V}' \neq \emptyset$ for all $y \in \mathcal{V}$.

Then, the operator \mathcal{A} is maximal monotone in $\mathcal{V} \times \mathcal{V}'$ and, since $D(\mathcal{A}) = \mathcal{V}$, we conclude as above that $0 \in R(\mathcal{A} + \mathcal{B})$, as desired.

Proposition 3.3 has the following immediate extension.

Proposition 4.4 *Under the above assumptions, let $\mathcal{T} : D(\mathcal{T}) \subset \mathcal{V} \rightarrow \mathcal{V}'$ be a maximal monotone operator (possibly multivalued) such that $\mathcal{B} + \mathcal{T}$ is maximal monotone on $\mathcal{V} \times \mathcal{V}'$. Then, for any $f \in \mathcal{V}'$, there is a unique solution y to the equation*

$$\mathcal{B}y + \mathcal{T}y + \mathcal{A}y = f. \quad (4.34)$$

Proof. Since $\mathcal{B} + \mathcal{T}$ and \mathcal{A} are maximal monotone operators in $\mathcal{V} \times \mathcal{V}'$ and \mathcal{A} is defined on all of \mathcal{V} , by the above mentioned Rockafellar result, $\mathcal{B} + \mathcal{T} + \mathcal{A}$ is maximal monotone on $\mathcal{V} \times \mathcal{V}'$ and so $R(\lambda F + \mathcal{B} + \mathcal{T} + \mathcal{A}) = \mathcal{V}'$. Then, letting $\lambda \rightarrow 0$ as in the above proof, we conclude that (4.34) has at least one solution $y \in D(\mathcal{B}) \cap D(\mathcal{T})$. The uniqueness follows as above. ■

In particular, Proposition 4.4 applies to the finite dimensional stochastic differential equation

$$\begin{aligned} dX + F(X)dt &\ni XdW, \quad t \in (0, T), \\ X(0) &= x, \end{aligned} \quad (4.35)$$

where $F : \mathbb{R} \rightarrow \mathbb{R}^d$ is a maximal monotone graph (multivalued) in $\mathbb{R}^d \times \mathbb{R}^d$, such that $D(F) = \mathbb{R}^d$. Then, the operator $\mathcal{T}y = e^{-W}F(e^W y)$ is maximal monotone in $\mathcal{H} \times \mathcal{H}$ (here, $H = V = V' = \mathbb{R}^d$) and $D(\mathcal{F}) = \mathcal{H}$. Hence $\mathcal{B} + \mathcal{F}$ is maximal monotone and so, for each $f \in \mathcal{H}$, equation (4.34) (and, implicitly, (4.35)) has a unique solution. This result is, in particular, applicable to the finite dimensional stochastic differential equations (4.35) with the monotonically nondecreasing, discontinuous function F after filling the jumps in discontinuity points.

Remark 4.5 Equation (2.1) with additive noise, that is, $dX + A(t)Xdt = dW$, $X(0) = x$, reduces via the transformation $X = Y - W$, to the random differential equation

$$\frac{dY}{dt} + A(t)(Y + W(t)) = 0, \quad Y(0) = x,$$

which, under assumptions (i)–(iii), has a unique solution $Y = Y(t, \omega)$ by virtue of the standard existence theory for the Cauchy problem associated with nonlinear, monotone and demicontinuous operators $Y \rightarrow A(t)(Y + W(t))$ from V to V' . (See [2], p. 183, [15].)

5 The subgradient case $A(t) = \partial\varphi(t, \cdot)$

Assume now that A satisfies (ii), (iii) and is the subdifferential of a continuous convex function on V . More precisely, $A(t) = \partial\varphi(t, \cdot)$, where $\varphi(t) = \varphi(t, \omega, y)$, is measurable in (t, ω) , continuous and convex in $y \in V$. That is,

$$A(t)y = \{\eta \in V'; \nu\langle\eta, y - z\rangle_V \geq \varphi(t, y) - \varphi(t, z), \\ \forall z \in V, t \in (0, T), \mathbb{P}\text{-a.s.}\}.$$

Then, \mathcal{A} defined by (4.10) is itself the subdifferential $\partial\Phi : \mathcal{V} \rightarrow \mathcal{V}'$ of the convex lower-semicontinuous function $\Phi : \mathcal{V} \rightarrow \mathbb{R}$ defined by

$$\Phi(y) = \mathbb{E} \int_0^T (\varphi(t, e^{W(t)}y(t)) - \nu|e^{W(t)}y(t)|_H^2)dt, \quad \forall y \in \mathcal{V}.$$

Then, taking into account that $\Phi^*(u) + \Phi(y) \geq \langle y, u \rangle_{\mathcal{V}'}$ for all $u \in \mathcal{V}'$, $y \in \mathcal{V}$, with equality if $u \in \partial\Phi(y)$, we may equivalently write equation (4.12) as

$$\mathcal{B}y + u = 0, \quad \Phi(y) + \Phi^*(u) - \nu\langle u, y \rangle_{\mathcal{V}} = 0, \quad (5.1)$$

where $\Phi^* : \mathcal{V}' \rightarrow \mathbb{R}$ is the conjugate of Φ , that is,

$$\Phi^*(v) = \sup\{\nu\langle v, u \rangle_{\mathcal{V}} - \Phi(u); u \in \mathcal{V}\}.$$

Taking into account that

$$\Phi(z) + \Phi^*(v) - \nu\langle v, z \rangle_{\mathcal{V}} \geq 0, \quad \forall v \in \mathcal{V}', z \in \mathcal{V},$$

it follows that the solution y to equation (4.12) (equivalently, (3.3)) is the solution to the minimization problem

$$\text{Min}\{\Phi(y) + \Phi^*(u) - \nu\langle u, y \rangle_{\mathcal{V}}; \mathcal{B}y + u = 0\}.$$

Equivalently,

$$\text{Min}\{\Phi(y) + \Phi^*(-\mathcal{B}y) + \nu\langle \mathcal{B}y, y \rangle_{\mathcal{V}}; y \in D(\mathcal{B})\}. \quad (5.2)$$

In this way, the Cauchy problem (3.3) and, implicitly, the stochastic differential equation (2.1) reduces to the convex minimization problem (5.2).

Taking into account that, under assumptions (i)–(iii), the function $y \rightarrow \Phi(y) + \Phi^*(-\mathcal{B}y) + \frac{1}{2} \mathbb{E}|e^{W(T)}y(T)|_H^2$ is convex, lower semicontinuous

and coercive on \mathcal{V} , we infer (without invoking Proposition 3.3) that (5.2) has a solution which turns out to be just the solution to (4.12). This might be an alternative way to prove existence and uniqueness for equation (4.12) in this special subgradient case.

This variational approach to (2.1) in the subgradient case $A = \partial\varphi$ inspired by the Brezis-Ekeland principle was already developed in [3]–[5] for some specific stochastic differential equations and it opens up the way to use convex analysis methods to stochastic differential equations.

6 Examples

Here, we briefly present a few classes of stochastic partial differential equations for which Theorem 3.1 is applicable. Everywhere in the following, W is the Wiener process (2.2) satisfying (2.8).

6.1 Nonlinear stochastic parabolic equations

Consider the stochastic equation in $\mathcal{O} \subset \mathbb{R}^d$

$$\begin{aligned} dX - \operatorname{div}(a(\nabla X))dt + \psi(X)dt &= XdW \text{ in } (0, T) \times \mathcal{O}, \\ X &= 0 \text{ on } (0, T) \times \partial\mathcal{O}, \quad X(0) = x \text{ in } \mathcal{O}. \end{aligned} \quad (6.1)$$

Here, $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a continuous and monotone mapping such that $a(0) = 0$ and

$$\begin{aligned} a(r) \cdot r &\geq a_1 |r|_d^p + a_2, \quad \forall r \in \mathbb{R}^d, \\ |a(r)|_d &\leq c_1 |r|_d^{p-1} + c_2, \quad \forall r \in \mathbb{R}^d, \end{aligned} \quad (6.2)$$

where $a_1, c_1 > 0$, $p > 1$.

The function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, monotonically nondecreasing, $0 = \psi(0)$ and

$$|\psi(r)| \leq C |r|_d^q, \quad \forall r \in \mathbb{R}. \quad (6.3)$$

Here, $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open subset with smooth boundary $\partial\mathcal{O}$ and $|\cdot|_d$ is the Euclidean norm of \mathbb{R}^d .

Consider the spaces $H = L^2(\mathcal{O})$, $V = W_0^{1,p}(\mathcal{O})$, $V' = W^{-1,p'}(\mathcal{O})$ and the operator $A : V \rightarrow V'$ defined by

$${}_{V'} \langle Ay, \varphi \rangle_V = \int_{\mathcal{O}} (a(\nabla y) \cdot \nabla \varphi + \psi(y)\varphi) d\xi, \quad \forall \varphi \in W_0^{1,p}(\mathcal{O}).$$

Under assumptions (6.2), (6.3), where

$$q < \frac{dp}{d-p} \text{ if } d > p; \quad q > p \text{ if } d < p,$$

by the Sobolev-Gagliardo-Nirenberg embedding theorem (see [1], [10]), it follows that A satisfies assumptions (i)–(iii), and so

Corollary 6.1 *Equation (6.1) has, for $x \in L^2(\mathcal{O})$, a unique solution $X \in L^p(0, T; W_0^{1,p}(\mathcal{O})) \cap C([0, T]; L^2(\mathcal{O}))$, \mathbb{P} -a.s. Moreover, $t \rightarrow e^{-W(t)}X(t)$ is $W^{-1,p'}(\mathcal{O})$ absolutely continuous on $[0, T]$, \mathbb{P} -a.s.*

Of course, the result remains true for the progressively measurable processes $a = a(t, r, \omega)$, $\psi = \psi(t, r, \omega)$, where $a(t, \cdot), \psi(t, \cdot)$ satisfy (6.2), (6.3) and the functions $t \rightarrow a(t, r)$, $t \rightarrow \psi(t, r)$ are of class L^∞ for each $r \in \mathbb{R}$.

Remark 6.2 The case $p = 1$, which was studied in [7], is not covered, however, by the present result. In fact, in this case, the space $V = W_0^{1,1}(\mathcal{O})$ is not reflexive and the solution X to (6.1) exists and is unique in a weak variational sense in the space of functions with bounded variation on \mathcal{O} .

Remark 6.3 Equation (6.1) with nonlinear boundary value conditions of the form

$$a(\nabla y) \cdot \vec{n} + \gamma(y) = 0 \quad \text{on } (0, T) \times \partial\mathcal{O},$$

where \vec{n} is the normal to $\partial\mathcal{O}$, and $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a monotonically increasing continuous function satisfying a growth condition (6.3), can be completely similarly treated in the variational setting $H = L^2(\mathcal{O})$, $V = W^{1,p}(\mathcal{O})$.

The Hölder continuity of $e^{-W}X$

Taking into account that $y = e^{-W}X$ is the solution to the random parabolic equation

$$\begin{aligned} y_t - e^{-W} \operatorname{div}(a(\nabla(e^W y))) + \mu y + e^{-W} \psi(e^W y) &= 0 \text{ in } Q_T = (0, T) \times \mathcal{O}, \\ y &= 0 \text{ on } (0, T) \times \partial\mathcal{O}, \\ y(0, \xi) &= x(\xi), \quad \xi \in \mathcal{O}, \end{aligned} \tag{6.4}$$

one can obtain from the regularity theory for parabolic quasi-linear equations with principal part in divergence form (see [14]) for $x \in L^\infty(\mathcal{O})$ Hölder

regularity results for solutions y to (6.4) and hence for solutions of (6.1). The result we obtain here is not the sharpest for (2.1) we can find via the rescaling approach, but it illustrates, however, the advantages of the method.

For simplicity, we assume that the above conditions on a and ψ are satisfied with $p = q = 2$.

We rewrite (6.4) as

$$\begin{aligned} y_t - \operatorname{div} \tilde{a}(t, \xi, y, \nabla y) + \tilde{a}_0(t, \xi, y, \nabla y) &= 0 \text{ in } (0, T) \times \mathcal{O}, \\ y &= 0 \text{ on } (0, T) \times \partial\mathcal{O}, \\ y(0, \xi) &= x(\xi), \quad \xi \in \mathcal{O}, \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} \tilde{a}(t, \xi, y, \eta) &= e^{-W(t, \xi)} a((\nabla W(t, \xi)y + \eta)e^{W(t, \xi)}) \\ \tilde{a}_0(t, \xi, \eta) &= -e^{-W(t, \xi)} \nabla W(t, \xi) \cdot a((\nabla W(t, \xi)y + \eta)e^{W(t, \xi)}) \\ &\quad + \mu(\xi)y + e^{-W(t, \xi)} \psi(e^{W(t, \xi)}y). \end{aligned}$$

Taking into account that, by (2.7)

$$\begin{aligned} e^{-|W(t, \xi)|} &\geq \gamma(\omega) > 0, \quad \forall (t, \xi) \in [0, T] \times \overline{\mathcal{O}}, \quad \omega \in \Omega, \\ e^{|W(t, \xi)|} &\leq \tilde{\gamma}(\omega) < \infty, \quad \forall (t, \xi) \in [0, T] \times \overline{\mathcal{O}}, \end{aligned}$$

where $\gamma, \tilde{\gamma} \in \bigcap_{0 < q < \infty} L^q(\Omega)$, we see that, for some $\gamma_1 = \gamma_i(\omega) \in \mathbb{R}$, $i = 1, \dots, 5$, $\gamma_1(\omega) > 0$,

$$\begin{aligned} \tilde{a}(t, \xi, y, \eta) \cdot \eta &\geq \gamma_1 |\eta|_d^2 - \gamma_2 |y|^2, \\ |(\tilde{a} + \tilde{a}_0)(t, \xi, y, \eta)|_d &\leq \gamma_3 |\eta|_d + \gamma_4 |y| + \gamma_5, \end{aligned}$$

for all $(t, \xi, y, \eta) \in [0, T] \times \overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^d$.

Now, since by Proposition 3.3 the solution y to (6.5) is in $C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H_0^1(\mathcal{O}))$, it follows by [14, Theorem 2.1, p. 425] that, for $x \in L^\infty(\mathcal{O})$, the solution y is bounded, i.e.,

$$|y(t, \xi)| \leq M, \quad \forall (t, \xi) \in (0, T) \times \mathcal{O}.$$

Hence, without loss of generality, we may replace in (6.5) the functions \tilde{a} and \tilde{a}_0 by

$$\tilde{a}_{(0)}(t, \xi, y, \eta) = \begin{cases} a_0(t, \xi, y, \eta) & \text{if } |y| \leq M, \\ a_0\left(t, \xi, M \frac{y}{|y|}, \eta\right) & \text{if } |y| > M, \end{cases}$$

and so, \bar{a} and \tilde{a}_0 satisfy Hypotheses (1.1)–(1.3) of Theorem 1.1 in [14], p. 19. Then, according to this theorem, for $x \in L^\infty(\mathcal{O})$, the solution y belongs to the space $H^{\alpha, \frac{\alpha}{2}}(Q_T)$ of Hölder continuous functions of order α in $\xi \in \mathcal{O}$ and order $\frac{\alpha}{2}$ in t on Q_T for some $\alpha > 0$.

We have, therefore,

Proposition 6.4 *Let $x \in L^\infty(\mathcal{O})$ and let a, ψ satisfy (6.2), (6.3) with $p = q = 2$. Then, the solution X to equation (6.1) satisfies*

$$e^{-W}X \in H^{\alpha, \frac{\alpha}{2}}(Q_T), \quad \mathbb{P}\text{-a.s.} \quad (6.6)$$

for some $\alpha = \alpha(\omega) \in (0, 1)$.

In particular, for each $t \in (0, T)$, $x \rightarrow X(t, \xi)$ is Hölder continuous.

Remark 6.5 By applying Theorem 4.1 in [14, p. 444], one can obtain for the solution y to (6.4), and implicitly for (6.1), L^∞ -estimates for the gradient ∇y . We omit the details.

Remark 6.6 Assume further that $a \in C^1(\overline{\mathcal{O}})$, $\psi \in C(\overline{\mathcal{O}})$ and that $x \in C(\overline{\mathcal{O}})$, $x = 0$ on $\partial\mathcal{O}$. Then, by Theorem 6.1 in [14], equation (6.5) has a unique solution $y \in H^{2+\gamma, 1+\frac{\gamma}{2}}([0, T] \times \overline{\mathcal{O}})$, where γ is the Hölder exponent of $W(t)$. Moreover, in this case, the solution y is continuous with respect to the initial data x from $C(\overline{\mathcal{O}})$ to $C([0, T]; L^2(\mathcal{O}))$. This implies that the solution X to (6.4) is pathwise continuous with respect to x .

6.2 The porous media equation

We first note that conditions (H1)–(H4) in [18] (see p. 56) imply our conditions (i)–(iii) on A . Therefore, all examples of stochastic partial differential equations from [18] (see Section 4.1 therein) are covered by our results provided the noise is linear multiplicative. Nevertheless, in this subsection we present an example from [18], namely, the stochastic porous media equation, in more detail.

Consider here the stochastic equation

$$\begin{aligned} dX - \Delta\psi(t, X)dt &= XdW && \text{in } (0, T) \times \mathcal{O}, \\ X(0) &= x && \text{in } \mathcal{O}, \\ \psi(X) &= 0 && \text{on } (0, T) \times \partial\mathcal{O}, \end{aligned} \quad (6.7)$$

where \mathcal{O} is a bounded domain in \mathbb{R}^d , $\psi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, monotonically increasing in r , there exist $a \in (0, \infty)$, $c \in [0, \infty)$, such that

$$\begin{aligned} r\psi(r) &\geq a|r|^p - c, \quad \forall r \in \mathbb{R}, \\ |\psi(r)| &\leq c(1 + |r|^{p-1}), \quad \forall r \in \mathbb{R}, \end{aligned} \tag{6.8}$$

where $p \in [\frac{2d}{d+2}, \infty)$ if $d \geq 3$, and $p \in (1, \infty)$, for $d = 1, 2$.

By the Sobolev-Gagliardo-Nirenberg embedding theorem, we have $L^p \subset H^{-1}(\mathcal{O})$. To write (6.7) in the form (2.1), we change in this case the pivot space H . Namely, we take $V = L^p(\mathcal{O})$, $H = H^{-1}(\mathcal{O})$, and V' is the dual of V with the pivot space $H^{-1}(\mathcal{O})$. We have, therefore, $V \subset H \subset V'$ and

$$V' = \{\theta \in \mathcal{D}'(\mathcal{O}); \theta = -\Delta v, v \in L^{p'}(\mathcal{O})\},$$

where Δ is taken in sense of distributions. The duality ${}_{V'} \langle \cdot, \cdot \rangle_V$ is defined as

$${}_{V'} \langle \theta, u \rangle_V = \int_{\mathcal{O}} \tilde{\theta} u d\xi, \quad \tilde{\theta} = (-\Delta)^{-1} \theta.$$

Δ is the Laplace operator with homogeneous Dirichlet boundary conditions, and so $\tilde{\theta} \in L^p(\mathcal{O})$.

The operator $A(t) : V \rightarrow V'$ is defined by

$${}_{V'} \langle A(t)y, v \rangle_V = \int_{\mathcal{O}} \psi(y) v d\xi, \quad \forall y, v \in V.$$

By (6.8), we infer that $A(t)$ satisfies (2.3), (2.4), that is,

$$\begin{aligned} {}_{V'} \langle A(t)y, y \rangle_V &\geq \alpha_1 |y|_V^p + \alpha_2, \quad \forall y \in V, \\ |A(t)y|_{V'} &\leq \gamma_1 |y|_V^{p-1} + \gamma_2, \quad \forall y \in V. \end{aligned}$$

(See [18, pp.71–72] for details.) It is also readily seen that $A(t) : V \rightarrow V'$ is demicontinuous.

Then, applying Theorem 3.1, we find that:

Corollary 6.7 *For each $x \in H^{-1}(\mathcal{O})$ there is a unique solution $X \in L^\infty(0, T; L^2(\Omega, H^{-1}))$ which satisfies $X \in L^p((0, T) \times \Omega; L^p(\mathcal{O}))$.*

Moreover, $t \rightarrow e^{-W(t)} X(t)$ is V' -absolutely continuous on $[0, T]$, \mathbb{P} -a.s. and

$$\mathbb{E} \int_0^T \left| e^{W(t)} \frac{d}{dt} (e^{-W(t)} X(t)) \right|_{V'}^{p'} dt < \infty. \tag{6.9}$$

Remark 6.8 In fact, through (6.9), Corollary 6.7 improves the corresponding results in [18], if the noise is linear multiplicative. In addition, Corollary 6.7 can be generalized. In fact, we can take $V = L_d \cap H^{-1}(\mathcal{O})$, where L_d is the Orlicz space on \mathcal{O} corresponding to a Δ_2 -regular d -function (see [1], p. 232). Also, if e_j are multipliers in $H^{-1}(\mathbb{R}^d)$, we can apply Theorem 3.1 to the case of the unbounded domain $\mathcal{O} = \mathbb{R}^d$. Then, by Theorem 3.1 we recover and improve many of the results of [19] and [8], provided the noise is linear multiplicative.

6.3 The stochastic transport equation

Consider the stochastic first order hyperbolic equation

$$\begin{aligned} dX(t, \xi) - \sum_{i=1}^d a_i(t, \xi) \frac{\partial X(t, \xi)}{\partial \xi_i} dt + b(t, \xi) X(t, \xi) dt \\ + \lambda |X(t, \xi)|^{p-2} X(t, \xi) dt = X(t, \xi) dW(t, \xi) \quad \text{in } (0, T) \times \mathcal{O}, \end{aligned} \quad (6.10)$$

$$X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O},$$

$$X(t) = 0 \quad \text{on } \Sigma = \left\{ (t, \xi) \in [0, T] \times \partial\mathcal{O}; \sum_{i=1}^d a_i(t, \xi) n_i(\xi) < 0 \right\},$$

where $\mathcal{O} \subset \mathbb{R}^d$ is, as usual, an open and bounded subset with smooth boundary $\partial\mathcal{O}$, $n = \{n_i\}_{i=1}^d$ is the normal to $\partial\mathcal{O}$ and $a_i, b : [0, T] \times \overline{\mathcal{O}} \rightarrow \mathbb{R}$, $i = 1, \dots, d$, are continuous functions with $\nabla_\xi a_i \in C([0, T] \times \overline{\mathcal{O}})$, $i = 1, \dots, d$. We assume also that $\lambda > 0$, $p \geq 2$, and that $x \in H^1(\mathcal{O}) \cap L^p(\mathcal{O})$, $x = 0$ on Σ .

By transformation (2.13), we reduce (6.10) to the random differential transport equation

$$\begin{aligned} \frac{\partial y}{\partial t} - \sum_{i=1}^d a_i e^{-W} \frac{\partial}{\partial \xi_i} (e^W y) + (b + \mu) y + \lambda e^{(p-2)W} |y|^{p-2} y = 0 \\ \text{in } (0, T) \times \mathcal{O}, \end{aligned} \quad (6.11)$$

$$y(0, \xi) = x(\xi), \quad \xi \in \mathcal{O}, \quad y = 0 \quad \text{on } \Sigma.$$

We consider now the spaces $\mathcal{V}, \mathcal{H}, \mathcal{V}'$ defined in Section 4, where $V = L^p(\mathcal{O})$, $H = L^2(\mathcal{O})$, $V' = L^{p'}(\mathcal{O})$ and set

$$\begin{aligned}\mathcal{A}y &= \lambda e^{(p-2)W} |y+x|^{p-2} (y+x) - \sum_{i=1}^d a_i e^{-W} \frac{\partial}{\partial \xi_i} (e^W x) + (b+\mu)x, \quad y \in \mathcal{V}, \\ \mathcal{B}y &= \frac{dy}{dt} + (\mu + \nu)y, \quad y \in D(\mathcal{B}),\end{aligned}$$

where we used the transformation $y \rightarrow y - x$. $D(\mathcal{B})$ is given by (4.11) with x replaced by 0, and ν by (2.8). Then (6.11) can be equivalently written as

$$\mathcal{B}y + \mathcal{T}_0(y) + \mathcal{A}y = 0,$$

where $\mathcal{T}_0 : \mathcal{V} \rightarrow \mathcal{V}'$ is given by

$$\begin{aligned}\mathcal{T}_0(y) &= - \sum_{i=1}^d a_i e^{-W} \frac{\partial}{\partial \xi_i} (e^W y) + (b - \nu)y, \quad \forall y \in D(\mathcal{T}_0), \\ D(\mathcal{T}_0) &= \{y \in \mathcal{V}; \ y = 0 \text{ on } \Sigma\}.\end{aligned}$$

We assume that

$$\frac{1}{2} \operatorname{div}_\xi a(t, \xi) + b(t, \xi) > \nu, \quad \forall (t, \xi) \in [0, T] \times \overline{\mathcal{O}}, \quad (6.12)$$

where $a = \{a_i\}_{i=1}^d$. Then, as is easily seen, \mathcal{T}_0 is monotone. It should be said that Proposition 4.4 is not directly applicable here because $\mathcal{T}_0 + \mathcal{B}$ is not maximal monotone. However, by [9] (see also [15, p. 345]), the operator $\mathcal{B} + \mathcal{T}_0$ is closable in $L^p((0, T) \times \mathcal{O})$ for fixed $\omega \in \Omega$ and its closure L is maximal monotone in $L^p((0, T) \times \mathcal{O}) \times L^{p'}((0, T) \times \mathcal{O})$. This implies that, for each fixed $\omega \in \Omega$, the equation

$$Ly + e^{-W} F(e^{-W} y) = 0 \quad \text{in } (0, T) \times \mathcal{O},$$

has a solution $y \in D(L)$ and so, arguing as in the proof of Lemma 4.2, it follows that the closure $\overline{\mathcal{B} + \mathcal{T}_0}$ of $\mathcal{B} + \mathcal{T}_0$ in $\mathcal{V} \times \mathcal{V}'$ is maximal monotone.

Then, by Proposition 4.4, the equation

$$\overline{\mathcal{B} + \mathcal{T}_0}(y) + \mathcal{A}y = 0$$

has a unique solution $y \in D(\overline{\mathcal{B} + \mathcal{T}_0})$. In other words, there is a sequence $\{y_n\} \subset D(\mathcal{B} + \mathcal{T}_0)$ such that $y_n \rightarrow y$ in \mathcal{V} and

$$(\mathcal{B} + \mathcal{T}_0)(y_n) + \mathcal{A}y_n \rightarrow 0 \quad \text{in } \mathcal{V}'. \quad (6.13)$$

We call such y a generalized solution to (6.11) and the corresponding $X = e^W y$ is a generalized solution to (6.10) in the above sense.

Remark 6.9 Similarly, one can treat stochastic equations of type (6.10) with an integral transport term $\int_{\tilde{\Omega}} K(t, x, z, z') X(t, z; z') dz'$ (see, e.g., [15], p. 346).

7 Extension to more general multiplicative noise

In this section, we indicate how the rescaling approach developed in the previous sections extends mutatis-mutandis to stochastic equations of the form

$$\begin{aligned} dX + A(t)X dt &= \sum_{k=1}^m \sigma_k(X) d\beta_k(t), \quad t \in [0, T], \\ X(0) &= x, \end{aligned} \quad (7.1)$$

where $\sigma_k : D(\sigma_k) \subset H \rightarrow H$ are linear generators of C_0 mutually commuting groups on H . In this case, (7.1) reduces to a random differential equation in the space H . (See [12], p. 203.) Namely, via the transformation $X = U(t)y$, where $U(t) = \prod_{k=1}^m e^{\sigma_k(\beta_k(t))}$, (2.1) reduces to the random differential equation

$$\begin{aligned} \frac{dy}{dt} + U^{-1}(t)A(t)(U(t)y) + \frac{1}{2} \sum_{k=1}^m U^{-1}(t)\sigma_k^2(U(t)y) &= 0, \quad t \in (0, T), \\ y(0) &= x. \end{aligned} \quad (7.2)$$

Here $e^{t\sigma_k}$ is the global flow on H generated by σ_k .

The existence in (7.2) follows as in the previous case, by taking

$$\mathcal{A}y = U^{-1}(t)A(t)(U(t)y)$$

and replacing \mathcal{H}, \mathcal{V} by the spaces of $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes y on $(0, T)$ such that

$$\mathbb{E} \int_0^T |U(t)y(t)|_H^2 dt < \infty \quad (\text{and, respectively, } \mathbb{E} \int_0^T |U(t)y(t)|_V^p dt < \infty).$$

We assume that U satisfies Hypothesis (iv) with $U(t)$ instead of $e^{W(t)}$ and take on \mathcal{H} the scalar product

$$\langle u, v \rangle = \mathbb{E} \int_0^T \langle U(t)u, U(t)v \rangle dt,$$

which extends to the duality pairing ${}_{\mathcal{V}'} \langle u, v \rangle_{\mathcal{V}}$ on $\mathcal{V} \times \mathcal{V}'$. Moreover, the space \mathcal{V} is reflexive and $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. If $A(t) : V \rightarrow V'$ satisfies assumptions (i)–(iii), then clearly $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ is monotone, demicontinuous and coercive, i.e., satisfies all properties of operator (4.10). If \mathcal{B} is defined as in (4.11), we rewrite (4.22) as $\mathcal{B}y + \mathcal{A}y = 0$. Moreover, Lemma 4.2 remains true in the present situation. The proof is completely similar, considering instead of (7.2) the equation

$$\begin{aligned} \frac{dy}{dt} + U^{-1}(t)J(U(t)y)|U(t)y|_V^{p-2} + (\mu + \nu)y &= f \\ y(0) &= x, \end{aligned}$$

for $f \in \mathcal{V}'$. Then we conclude, as in the proof of Proposition 3.3, that the range of $\mathcal{B} + \mathcal{A}$ is all of \mathcal{V}' and, in particular, that equation (7.2) has a unique $(\mathcal{F}_t)_{t \geq 0}$ -adapted solution $y : [0, T] \rightarrow H$, such that

$$\mathbb{E} \sup_{t \in [0, T]} |U(t)y(t)|_H^2 < \infty \quad (7.3)$$

$$\mathbb{E} \int_0^T \left| U(t) \frac{dy}{dt} \right|_{V'}^{p'} dt < \infty. \quad (7.4)$$

Then, Theorem 3.1 remains valid in the present case with $U(t)$ instead of $e^{W(t)}$.

Consider as an example the nonlinear diffusion equation (see [5], [11])

$$\begin{aligned} dX - \operatorname{div}(a(\nabla X))dt - \frac{1}{2} b \cdot \nabla(b \cdot \nabla X)dt &= b \cdot \nabla X d\beta, \\ t \in [0, T], \quad \xi \in \mathcal{O}, \end{aligned} \quad (7.5)$$

$$X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O}; \quad X = 0 \text{ on } (0, T) \times \partial\mathcal{O}.$$

Here, $\mathcal{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded open domain with smooth boundary, β is a Brownian motion, $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a monotone mapping satisfying (6.2) and $b \in (C^1(\overline{\mathcal{O}}))^d$ satisfies: $\operatorname{div} b = 0$, $b \cdot \nu = 0$ on $\partial\mathcal{O}$ (ν is the normal to $\partial\mathcal{O}$). It should be mentioned that (7.5) is equivalent to the Stratonovich stochastic equation

$$\begin{aligned} dX - \operatorname{div}(a(\nabla X))dt &= (b \cdot \nabla X) \circ d\beta \text{ in } (0, T) \times \mathcal{O}, \\ X(0) &= x, \quad X = 0 \text{ on } (0, T) \times \partial\mathcal{O}. \end{aligned}$$

Equation (7.5) is of the form (7.1), where $H = L^2(\mathcal{O})$, $V = W_0^{1,p}(\mathcal{O})$, $V' = W^{1,p'}(\mathcal{O})$, and

$$(U(t)f)(\xi) = \exp(\beta(t)b)(f)(\xi) = f(Z(\beta(t), \xi)), \quad t \geq 0, \quad \xi \in \mathcal{O},$$

where $Z = Z(s)$ is the flow of diffeomorphisms on $\overline{\mathcal{O}}$ generated by the Cauchy problem

$$\frac{dZ}{dt} = b(Z), \quad s \geq 0; \quad Z(0) = \xi \in \overline{\mathcal{O}}.$$

(See [5], [11].) Then, (7.2) is, in this case,

$$\begin{aligned} \frac{\partial y}{\partial t}(t, \xi) - U^{-1}(t)(\operatorname{div}(a(\nabla_\xi(U(t)y(t)))))) &= 0, \quad t \in [0, T], \quad \xi \in \mathcal{O}, \\ y(0, \xi) &= x(\xi), \quad y = 0 \text{ on } (0, T) \times \partial\mathcal{O}, \end{aligned} \quad (7.6)$$

and, as seen above, it follows the existence of a solution $y : [0, T] \rightarrow L^2(\mathcal{O})$ satisfying (7.3)–(7.4). More details will be contained in forthcoming work.

8 Appendix

Lemma 8.1

- (j) *Let $y = y(t)$, $t \in [0, T]$, be an H -valued $(\mathcal{F}_t)_{t \geq 0}$ -adapted process with continuous sample paths satisfying (3.4)–(3.7). Then, $X = e^W y$ is a strong solution to (2.1), which satisfies (2.10)–(2.12).*
- (jj) *Let $X = X(t)$, $t \in [0, T]$, be an H -valued $(\mathcal{F}_t)_{t \geq 0}$ -adapted process with continuous sample paths satisfying (2.10)–(2.12). Then, $y = e^{-W} X$ satisfies (3.4)–(3.7).*
- (jjj) *Let $y \in D(\mathcal{B})$ (see (4.11)). Then $t \mapsto y(t)e^{W(t)} \in H$ is continuous \mathbb{P} -a.s. and, $\forall t \in [0, T]$,*

$$y(t)e^{W(t)} = y(0) + \int_0^t \left[e^{W(s)} \frac{dy}{ds} + \mu e^{W(s)} y(s) \right] ds + \int_0^t y(s) e^{W(s)} dW(s).$$

Proof. We shall follow an argument from [7]. We first note that, by the last item in (2.6), it follows that with y also $t \mapsto e^{W(t)} y(t) \in H$ is continuous. We shall only prove (j). (jj) and (jjj) follow by the same arguments. If $\{f_j\}$

is an orthonormal basis in H such that $f_j \in V, \forall j$, we have, for each $\varphi \in V, t \in [0, T]$,

$$\langle \varphi, e^{W(t)} y(t) \rangle = \sum_{j=1}^{\infty} \langle e^{W(t)} \varphi, f_j \rangle \langle f_j, y(t) \rangle$$

and so, by Itô's formula, we have

$$e^{W(t, \xi)} = 1 + \int_0^t e^{W(s, \xi)} dW(s, \xi) + \mu(\xi) \int_0^t W(s, \xi) ds, \quad \forall t \in [0, T], \quad \forall \xi \in \mathcal{O}.$$

This yields, via the stochastic Fubini theorem,

$$\begin{aligned} \langle e^{W(t)} \varphi, f_j \rangle &= \langle \varphi, f_j \rangle + \sum_{k=1}^m \left\langle \varphi e_k f_j, \int_0^t e^{W(s, \zeta)} d\beta_k(s) \right\rangle + \int_0^t \langle \mu e^{W(s)} \varphi, f_j \rangle ds \\ &= \langle \varphi, f_j \rangle + \sum_{k=1}^{\infty} \int_0^t \langle e_k e^{W(s)} \varphi, f_j \rangle d\beta_k(s) + \int_0^t \langle \mu e^{W(s)} \varphi, f_j \rangle ds. \end{aligned}$$

Here, $y_\varepsilon = J_\varepsilon(y)$ and J_ε is a mollifier operator $J_\varepsilon \in L(V', H)$ chosen in a such a way that $\lim_{\varepsilon \rightarrow 0} J_\varepsilon(y) = y$ in V' . (Such a family of mappings J_ε always exists.) Then, we see by (3.6) that

$$\begin{aligned} \frac{dy_\varepsilon}{dt} + J_\varepsilon(e^{-W} \eta) + J_\varepsilon(\mu y) &= 0, \quad \text{a.e. } t \in (0, T), \\ y_\varepsilon(0) &= J_\varepsilon(x), \end{aligned} \tag{8.1}$$

where $\eta = A(t)(e^{W(t)} y(t))$. Since $t \rightarrow \langle f_j, y_\varepsilon(t) \rangle$ is absolutely continuous, we can apply the Itô product rule to get, by (8.1), that

$$\begin{aligned} \langle e^{W(t)} \varphi, f_j \rangle \langle f_j, y_\varepsilon(t) \rangle &= \langle \varphi, f_j \rangle_H \langle f_j, x \rangle \\ &\quad - \int_0^t \langle e^{W(s)} \varphi, f_j \rangle \langle f_j, J_\varepsilon(e^{-W(s)} \eta(s) + \mu y(s)) \rangle ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \langle f_j, y_\varepsilon(s) \rangle \langle e_k e^{W(s)} \varphi, f_j \rangle d\beta_k(s) \\ &\quad + \int_0^t \langle f_j, y_\varepsilon(s) \rangle_H \langle \mu e^{W(s)} \varphi, f_j \rangle_H ds, \quad \forall j \in \mathbb{N}, \end{aligned}$$

where $\eta = Ay$. Summing up and interchanging the sum with the integrals, we obtain \mathbb{P} -a.s., for all $t \in [0, T]$ and all $\varphi \in V$,

$$\begin{aligned} \langle \varphi, e^{W(t)} y_\varepsilon(t) \rangle &= \langle e^{W(s)} \varphi, x \rangle - \int_0^t \langle e^{W(s)} \varphi, J_\varepsilon(e^{-W(s)} \eta(s) + \mu y(s)) \rangle ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \langle \varphi, e_k e^{W(s)} y_\varepsilon(s) \rangle d\beta_k(s) + \int_0^t \langle \mu e^{W(s)} \varphi(s), y_\varepsilon(s) \rangle ds. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we see that $X = e^W y$ satisfies (2.11). It is also clear that X satisfies all the conditions in Definition 2.1. ■

Lemma 8.2 *Let $1 < p < 2$ and $y \in L^p(0, T; V) \cap L^2(0, T; H)$ such that $\frac{dy}{dt} \in L^{p'}(0, T; V') + L^2(0, T; H)$. Then $y \in C([0, T]; H)$ and*

$$\frac{1}{2} |y(t)|_H^2 = \frac{1}{2} |x|_H^2 + \int_0^t \left\langle \frac{dy}{ds}, y \right\rangle ds, \quad \forall t \in [0, T].$$

Proof. We have ($\frac{dy}{dt}$ is taken in $\mathcal{D}'(0, T)$)

$$\frac{dy}{dt} = f_1 + f_2, \quad f_1 \in L^{p'}(0, T; V'), \quad f_2 \in L^2(0, T; H).$$

We consider a mollifier $\rho_\varepsilon \in C^\infty$ and set

$$y_\varepsilon(t) = (y * \rho_\varepsilon)(t), \quad f_\varepsilon^1 = f_1 * \rho_\varepsilon, \quad f_\varepsilon^2 = f_2 * \rho_\varepsilon.$$

We have $y_\varepsilon \in C^1([0, T]; V)$ and $\frac{dy_\varepsilon}{dt} = f_\varepsilon^1 + f_\varepsilon^2$. This yields

$$\begin{aligned} \frac{1}{2} |y_\varepsilon(t) - y_{\varepsilon'}(t)|_H^2 &= \frac{1}{2} |x|_H^2 + \int_0^t \langle f_\varepsilon^1(s) - f_{\varepsilon'}^1(s), y_\varepsilon(s) - y_{\varepsilon'}(s) \rangle ds \\ &\quad + \int_0^t \langle f_\varepsilon^2(s) - f_{\varepsilon'}^2(s), y_\varepsilon(s) - y_{\varepsilon'}(s) \rangle ds. \end{aligned}$$

Since

$$\begin{aligned} f_\varepsilon^1 &\longrightarrow f^1 && \text{in } L^{p'}(0, T; V'), \\ f_\varepsilon^2 &\longrightarrow f^2 && \text{in } L^2(0, T; H), \\ y_\varepsilon &\longrightarrow y && \text{in } L^p(0, T; V) \cap L^2(0, T; H), \end{aligned}$$

as $\varepsilon \rightarrow 0$, we get that

$$y_\varepsilon \longrightarrow y \quad \text{in } C([0, T]; H) \text{ as } \varepsilon \rightarrow 0,$$

and that

$$\frac{1}{2} |y(t)|_H^2 = \frac{1}{2} |x|_H^2 + \int_0^t \langle f^1 + f^2, y \rangle ds, \quad \forall t \in [0, T],$$

as claimed. ■

Lemma 8.3 *Let $G(z) = J(z)|z|_V^{p-2}$ and $\{z_n\} \subset L^p((0, T) \times \Omega; V)$ be such that $z_n \rightarrow z$ weakly in $L^p((0, T) \times \Omega; V)$ and $G(z_n) \rightarrow \eta$ weakly in $L^{p'}((0, T) \times \Omega; V')$. Assume that*

$$\limsup_{n \rightarrow \infty} \mathbb{E} \int_0^T (T-t) \langle G(z_n(t)), z_n(t) \rangle dt \leq \mathbb{E} \int_0^T (T-t) \langle \eta(t), z(t) \rangle dt. \quad (8.2)$$

Then $\eta = G(z)$ a.e. in $(0, T) \times \Omega$.

Proof. We set $\Phi(z) = \frac{1}{p} |z|_V^p$ and note that Φ is Gâteaux differentiable and $\nabla \Phi = G$. Since Φ is convex and continuous on V , we have by Fatou's lemma

$$\liminf_{n \rightarrow \infty} \mathbb{E} \int_0^T (T-t) \Phi(z_n(t)) dt \geq \mathbb{E} \int_0^T (T-t) \Phi(z(t)) dt. \quad (8.3)$$

We also have, for $u \in L^p((0, T) \times \Omega; V)$,

$$\mathbb{E} \int_0^T (T-t) \langle G(z_n(t)), z_n(t) - u(t) \rangle dt \geq \mathbb{E} \int_0^T (T-t) (\Phi(z_n(t)) - \Phi(u(t))) dt.$$

Then, letting $n \rightarrow \infty$, we get by (8.2)–(8.3)

$$\mathbb{E} \int_0^T (T-t) (\Phi(z(t)) - \Phi(u(t))) dt \leq \mathbb{E} \int_0^T (T-t) \langle \eta(t), z(t) - u(t) \rangle dt.$$

Taking $u = z + \lambda v$, $v \in L^p((0, T) \times \Omega; V)$, dividing by λ and letting $\lambda \rightarrow 0$, we obtain, since $\nabla \Phi = G$,

$$\mathbb{E} \int_0^T (T-t) \langle G(z(t)) - \eta(t), v(t) \rangle dt \leq 0, \quad \forall v \in L^p((0, T) \times \Omega; V),$$

where we used the elementary inequality

$$\frac{1}{p} a^p - \frac{1}{p} b^p \leq \max(a^{p-1}, b^{p-1}) |a - b|, \quad a, b \in [0, \infty),$$

to justify the interchange of $\lim_{\lambda \rightarrow 0}$ with the integrals. Hence $\eta = G$ a.e. in $(0, T) \times \Omega$, as claimed. ■

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